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Source: *The Journal of Finance*, Vol. 43, No. 4 (Sep., 1988), pp. 893-911
Published by: Wiley for the American Finance Association
Stable URL: https://www.jstor.org/stable/2328142

THE JOURNAL OF FINANCE • VOL. XLIII, NO. 4 • SEPTEMBER 1988

Term Structure Multiplicity and Clientele in Markets with Transactions Costs and Taxes

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ABSTRACT

The authors investigate term structure with realistic transactions costs and taxes. Its properties are derived from a certain no-arbitrage condition via duality theory in convex programming. Transactions costs imply an infinite multiplicity of term structures. A simple example with realistic transactions costs shows that this multiplicity can induce a valuation range of over 277 basis points. Transactions costs also allow equilibrium without short sale restrictions. The authors find the minimum transactions costs that prevent arbitrage. In addition, the exact conditions for weak clientele, in which investors will not buy some bonds and may not sell any that they already hold, are established.

MOST RESEARCH ON THE pricing of riskless bonds (hereafter bonds) uses a no-arbitrage model. Equilibrium pricing is based on the assumption that there is no buy-and-hold investment strategy that provides riskless arbitrage profit.[1] However, most of these studies ignore transactions costs.[2]

The clientele model (hereafter SS) developed by Stephen Schaefer [19] adopts a no-arbitrage assumption, NA, equivalent to the existence of a valuation operator. This operator is a vector of discount factors known as the term structure. SS shows that NA implies the existence of a tax-bracket-specific term structure that ensures that the bonds' after-tax present values do not exceed their prices. SS also shows that, in markets with spanning, (positive) term structures consistent with NA will not exist for some brackets. (See his example in Subsection IV*A* below.) Investors in such brackets would have arbitrage opportunities.

To resolve this NA violation and make equilibrium feasible, SS assumes a prohibition on shorting. This implies clientele effects in the usual sense; i.e., investors in some brackets hold some but not all bonds. There may be, however, a bracket in which investors hold positive amounts of all bonds. This would be a

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[1] An exception is the recent study by Constantinides and Ingersoll [2], which allows dynamic strategy.

[2] Some effects of linear transactions costs on the valuation of risky securities were studied by Garman and Ohlson [8]. However, these effects are of a different nature from those studied here. Schaefer [17, 18], Hodges and Schaefer [10], and Ronn [15] considered bid-ask spreads, but all except Ronn prohibited short sales and all sought a unique term structure for each tax class. The purpose of this paper is to disprove such uniqueness.

"representative" or "marginal" investor's bracket in which NA holds without the shorting prohibition.

Meyers [11] explored the relation between this pricing operator, which is equivalent to NA, and the investor's marginal utility for future cash in a perfect market. Ross [16] did this in markets with taxes.

The first major difference between this paper and most of the literature on term structure is the way in which taxes are modeled. The studies on term structure estimation with taxes implicitly assume that an investor's tax bracket captures all properties that we attribute to his or her current position. Such capture is accomplished by formulating NA in terms of a fixed after-tax payment matrix. This, in turn, restricts the tax function to a particular linear form. However, our model is not restricted in this way because it uses a pre-tax payment matrix. This allows a separate tax function that can be piecewise linear and convex.[3]

The second difference is the treatment of transactions costs. This paper generalizes the models of Prisman [12] and Ross [16] by including transactions costs. This inclusion allows the possibility of equilibria in which there are no short-sales prohibitions. The market frictions in this model are the pair of both the transactions-cost schedule and the tax schedule facing the investor in question. Each investor class is defined by the distinct schedule of frictions (i.e., transactions costs and taxes) it faces, and this schedule depends on the investor's previously acquired position. Thus, all untaxed investors holding no previous position would be a class, and the only investor holding a particular set of securities long and a particular set short would be a class of one.

Our assumptions allow a distinct piecewise-linear and convex schedule of transactions costs paid currently for each bond traded: bid-ask spread and cost of borrowing a bond for shorting. In particular, this allows zero short-borrowing costs up to the amount, if any, of the investor's previous long position in the bond shorted. It also allows limitation of short selling, i.e, an infinite borrowing cost for all shorting beyond some level. (If this level is zero, we have Schaefer's prohibition on shorting.) Taxes can have any flat rate, or progressive rates in brackets, under one hundred percent for each type of income subject to a distinct tax treatment.[4]

These assumptions imply that, when one formulates the no-arbitrage condition for an investor class as an optimization problem, it is a convex problem. This simplifies things by allowing one to appeal to the standard results of duality

[3] To be correct, we should say affine instead of linear. An affine function differs from a linear function in that it does not necessarily intersect the origin. That a function $f(\cdot)$ is linear means that $f(ax + by) = af(x) + bf(y)$ for all constants a and b and all x and y in the domain of $f(\cdot)$. Setting $a = 0$ and $b = 0$ thus implies that $f(0) = 0$; i.e., any linear function $f(\cdot)$ intersects the origin.

[4] Tax laws in most countries do not tax arbitrage profits that are in the form of current cash since the individual long and short trades that provide this profit represent the exchange of current cash for bonds of equal value. The investor's tax flow is affected as the future cash stream or amortization of each bond traded provides interest income or expense and/or capital gain or loss. Thus, all taxes are paid in the future, in contrast with all transactions costs (for the buy-and-hold change of positions modeled here), which are paid currently.

theory in convex programming in order to characterize term structures in the presence of frictions.[5]

Realistic frictions always have a kink (nondifferentiability) at zero transactions. We use the duality approach to show that this kink is a second source of term-structure multiplicity in addition to the obvious source—the marginal after-tax payment matrix having rank less than the number of payment dates. This kink allows us to prove our main result, that either of two alternative no-arbitrage assumptions implies that there is an uncountably infinite, convex, and compact multiplicity (set), say D, of no-arbitrage term structures for each investor class. Furthermore, D is "fat," i.e., has a nonempty interior. The implicit assumption of term-structure uniqueness pervades the literature; its unnoticed but crucial role suggests the potential significance of this multiplicity. Establishing this multiplicity result is intended to correct misimpressions and to provide a basis for further research.

For example, this multiplicity allows Dermody and Rockafellar [3, 4, 5, 6] to show that each future cash flow, w, has a lower-value bound and an upper-value bound. This lower-value bound is the most current cash that can be extracted from the market by trading when w is used as a subsidy to future cash flow; the upper-value bound is the current cost of replicating w in the market by trading. Because D is fat, these two values are obtained with different term structures (or disjoint sets of term structures) within D associated with the upper- and lower-value bounds. Different w may have different pairs of such term structures within D. Thus, the upper and lower values are nonlinear in w. From an investor's point of view, two different w at given prices might then be unattractive separately but attractive together. This nonadditivity is exploited in [3] to show examples of serious valuation errors by the standard NPV method currently used in financial markets.

The multiplicity result leads immediately to our minimum transactions-cost and clientele results. Both highlight the different nature of a model with transactions costs and the consequent different type of clientele effect, called weak clientele. The former result characterizes, in terms of the market data, the minimum transactions costs necessary to prevent arbitrage. Our complete no-arbitrage assumption implies that, for each bond in the market, there is at least one investor who exhibits a clientele effect for that bond in the usual sense (will not buy the bond and will sell any he or she already acquired). In general, other investors do not exhibit this effect.

Section I introduces the mathematical perspective and tools used in the analysis. Section II specifies the model. Section III is the multiplicity analysis. It shows how the endogenous term structures of our model are characterized in terms of the exogenous circumstances, particularly the short-borrowing costs. Section IV introduces typical transactions costs into Schaefer's original example. This introduction changes the number of equilibrium price vectors and associated no-arbitrage term structures from the empty set to an infinite set. The set of no-arbitrage term structures has a 277 basis-point range for a taxed investor in this

[5] These standard results in convex programming are found in Geoffrion [9], Avriel [1], and Rockafellar [14], in increasing order of difficulty and generality.

example. Sections V and VI characterize the minimum transactions costs that prevent arbitrage and the consequent clientele effects, respectively.

I. Preliminaries

A. Linear vs. Nonlinear Frictions and Position vs. Change of Position

Introducing transactions costs makes it more compelling to model the *frictions* (i.e., *transactions-cost and tax schedules*) as nonlinear and specific to each investor class. There is a fundamental difference between markets with common linear frictions (including no friction), on the one hand, and markets with nonlinear frictions and/or frictions that depend on an investor's circumstances (e.g., tax status and bond position), on the other hand. They differ in the most convenient representation of the investor's-choice variable as well as in the nature of arbitrage and clientele.

If arbitrage existed in the former case, every investor would exploit it with an infinite position. Thus, arbitrage is inconsistent with equilibrium. In the latter case, arbitrage may depend on the investor's position; there may be positions held by no one from which finite arbitrage would be available in equilibrium. Nonlinear frictions may make arbitrage finite.

These implications suggest that a no-arbitrage assumption, NA, in the former case be modeled differently from NA in the latter. In our model, frictions are nonlinear and depend on an investor's individual circumstances. NA in our model means that no change in *any* investor's buy-and-hold position provides positive arbitrage. Thus, if NA holds, we have a separate condition for every investor class, i.e., every distinct set of frictions facing at least one investor. Each set of frictions provides a separate set of no-arbitrage term structures, D. This emphasis on change of position (as the investor's choice variable) contrasts with the emphasis on position in the previous literature. (See Ross [16].) Consider an investor holding any position, y, in riskless bonds and facing any set of frictions. In our model, NA means that no arbitrage, net of frictions, is available by changing from y to any other position $(y + x)$. This x is the investor's optimization instrument. Such NA provides the properties that characterize term structures in our model.

The frictions treated in this paper present us with a clientele effect different from the usual type. A weak long clientele effect for a bond means that an investor will not buy the bond, but neither will he or she necessarily short it (even if he or she has previously acquired it and faces no short-borrowing cost). Weak short clientele is symmetrically defined. This contrasts with the usual clientele effect: an investor will not buy a bond and will short it if not prohibited, and in particular will sell the bond if he or she previously acquired it.

B. Three Gradations of NA Assumptions

We use both the weak and the strong forms of NA proposed by Garman [7] and the *complete no-arbitrage*, CNA, proposed by Dermody and Rockafellar [4].

We explain these three assumptions here and define them mathematically in the next subsection.

Weak no-arbitrage, WNA, means there is no change, x, of any investor's position that provides positive current net cash flow without requiring a net cash payment from him or her in some future period. Current cash flow is the money received from the investor's short position minus that required to buy his or her long position, net of his or her transactions costs. Future cash flow is the payments received from the investor's long position minus those required to fulfill his or her short position, net of his or her taxes. *Strong no-arbitrage*, SNA, is the same except that it also precludes positive net cash flow in the future, not just currently.

CNA is the assumption that SNA holds and that there is no nonzero change of position x for any investor that provides him or her exactly zero change in cash flows, at all times, current and future. WNA means there is no free lunch in the form of current cash, and SNA means there is no free lunch in the form of cash received at any time. CNA implies no free lunch at any time and no free ride.

In most of our analysis, we use WNA, but we assume SNA or CNA at the end of Section III in the multiplicity result. NA conditions are properties of the market and depend on the particular frictions facing each investor.

C. Duality

The no-arbitrage assumptions, NA, can be defined in terms of the mathematical program (PP) below:

$$\max_{x} \{\Pi(x)\} \quad \text{subject to } \Gamma(x) \geq \mathbf{0}. \tag{PP}$$

This program describes maximizing arbitrage profit by a particular investor whose transactions costs are captured in his or her current-cash flow (scalar-valued) function, $\Pi(\cdot)$ and whose future tax schedule is captured in his or her future cash flow (n-vector-valued) function, $\Gamma(\cdot)$.[6]

This maximization is done over the investor's choice, $x = (x_1, \cdots, x_m)$, of his or her change in position in the m riskless bonds in the market. Both $\Pi(\cdot)$ and $\Gamma(\cdot)$ are functions of x. The constraint constant, $\mathbf{0}$, and the change of position, x, are n- and m-vectors, respectively. Here, $\Pi(\cdot)$ is the current (scalar) proceeds from the change of position, x, e.g., payment received from the short position (from negative x_i values) minus the cost of the long position (of positive x_i values), net of transactions costs. The function $\Gamma(\cdot) = [\Gamma_1(\cdot), \cdots, \Gamma_n(\cdot)]$ is the n-vector of changes in each of the n future periods' after-tax cash flow stemming from x

[6] If an investor is already long in bond i and shorts it, then his or her short-borrowing cost is zero up to the amount of his or her long position. The investor receives the bid price, $-x_i p_i - b(x_i)$, for $x_i < 0$, where $b(x_i)$ reflects only the bid-ask spread for x, up to the amount of his or her short position. If an investor is already short in bond i and longs it, then he or she does so by closing out part or all of the short position. The investor does so by buying the bond at the ask price and selling the remaining short-borrowing rights until maturity that he or she acquired when he or she previously took the short position. The investor pays the ask price minus the short-borrowing cost, $xp + b(x)$, for $x_i > 0$, where $b(x_i)$ reflects the bid-ask spread and the negative of the short-borrowing cost.

(e.g., cash flow from the long position minus that from the short position, net of future taxes). In Subsection II*B*, we show that our assumptions imply that $\Pi(\cdot)$ and $\Gamma(\cdot)$ are concave functions of x.

Definitions: *The weak no-arbitrage assumption*, WNA, means that the optimal value of (PP) is zero. *The strong no-arbitrage assumption*, SNA, means that WNA holds and that all optimal solutions satisfy the constraint as an equality. *The complete no-arbitrage assumption*, CNA, means that SNA holds and that $x = 0$ is the unique optimal solution of (PP).

Here, $\Pi(\cdot)$ and $\Gamma(\cdot)$ depend on the frictions facing the investor, which may depend on his or her circumstances, including his or her initial position, y. Note that the optimal value of (PP) is nonnegative because, in our model, no trade, $x = 0$, implies no change in cash flow at any time, i.e., that $\Pi(\mathbf{0}) = 0$ and $\Gamma(\mathbf{0}) = 0$, which satisfies the constraint and gives the objective of (PP) a value of zero. Since $x = \mathbf{0}$ is thus feasible, $x^* = \mathbf{0}$ is an optimal solution of (PP) when WNA holds. The optimization problem (DD) is the dual problem to (PP), in which d is the n-vector of Lagrange multipliers for the constraint function, $\Gamma(\cdot)$, and can be interpreted as the term structure:

$$\min_{d\geq 0}\left\{\max_{x}\left\{\Pi(x) - \sum_{j=1}^{n} \Gamma_j(x)\, d_j\right\}\right\}. \tag{DD}$$

(See Geoffrion [9].) This $d = (d_1, \cdots, d_n)$ is a vector of (nonnegative) shadow prices of the n future nonnegative-cash flow constraints, $\Gamma_i(x) \geq 0$, in (PP). Each d_j is the discount factor for period j. Thus, a nonnegative d solving (DD) is a term structure for an investor facing the frictions (i.e., transactions cost and tax schedules) captured in $\Pi(\cdot)$ and $\Gamma(\cdot)$.

Problem (DD) is the process of choosing the d-value, d^*, from among all $d \geq \mathbf{0}$, which minimizes a particular quantity: the maximum possible present value of the investor's current and future after-tax changes in cash flow stemming from x. There is an x such that $\Gamma(x) > \mathbf{0}$, e.g., a portfolio of one of each bond. This ensures via duality theory that, if $\Pi(\cdot)$ and $\Gamma(\cdot)$ are convex (as our assumptions imply), then there is an optimal solution to (DD), d^*, and the optimum value of (DD) equals that of (PP). (See Theorem 3 in Geoffrion [9].) Thus, WNA implies that there is at least one nonnegative d^* solving (DD) and that the optimal value of (DD) vanishes. Such d^* are the term structures, which are the subject of our paper. They depend on properties of $\Pi(\cdot)$ and $\Gamma(\cdot)$, especially the frictions captured in them.

D. Subgradients

A subgradient of any function, $F(\cdot)$, is a generalization of the gradient, i.e., of the vector of partial derivatives of $F(\cdot)$. A subgradient of $F(\cdot)$ at a point $\bar{x}$ in its domain is the slope of any linear function (whose graph is a hyperplane of dimension one less than that of the range of $F(\cdot)$) supporting the graph of $F(\cdot)$ at $\bar{x}$, i.e., any such linear function that is coincident with the graph of $F(\cdot)$ at $\bar{x}$

and that is on or below $F(\cdot)$ for all x.[7] See the subgradients of $F(\cdot)$ at x° and at x'' in Figure 1.

Definition: Let $F(\cdot)$ be a function defined on R^n. A subgradient of $F(\cdot)$ at $\bar{x}$ is a vector, $\omega \in R^n$, for which $F(x) \geq F(\bar{x}) + \omega(x - \bar{x})$ for every x.

If F is a convex function and $\bar{x}$ is in the relative interior of the domain of $F(\cdot)$, then a subgradient at $\bar{x}$ exists. It is unique if and only if $F(\cdot)$ is differentiable at $\bar{x}$. At such $\bar{x}$ (e.g., at x° in Figure 1), the subgradient is the gradient. When F is kinked (nondifferentiable) at $\bar{x}$, there is an infinite, closed, and convex set of subgradients there. $F(\cdot)$ has a set of about 90° of subgradients at x'' in Figure 1.

II. Model

A. Notation

The exogenous circumstances are these: the pre-tax payment matrix of bonds' cash flows, A; the vector of prices, p, of the bonds traded in the market; the schedule $T(\cdot)$, of changes in future tax payments of an investor in the class due to his or her change in position, x; and the transactions cost schedule $b(\cdot)$. Both $b(\cdot)$ and $T(\cdot)$ are functions of x. Different investor classes can have different tax schedules (including exemptions and exhaustible tax rebates for losses) as well as different $b(\cdot)$, but all face the same A and p.

We list some of our notational conventions here. Each riskless bond is denoted by $i = 1, \cdots, m$ and each future payment date by $j = 1, \cdots, n$. Note that ":=" means "is defined as." For any vectors, v and s, we write $v > s$ to mean $v_k > s_k$ for all k. Bold "0" is a vector $(0, \cdots, 0)$.

$A := [a_{ij}]$, which is the $m \times n$ matrix of future bond payments before tax, with each bond i represented by row i of A. a_{ij} is the payment of bond i on future payment date j.

$x := (x_1, \cdots, x_m)$, which is the row vector of change in an investor's buy-and-hold position denominated in number of bonds. $x_i > 0$ is a long purchase of bond i, and $x_i < 0$ is a short sale of bond i. x^* is an optimal solution to the arbitrage-maximization problem (e.g., of (P) or (PP)).

$b(\cdot)$ is the transactions-cost (scalar) function for x, e.g., bid-ask spread and short-borrowing cost. It is paid currently. $b(\mathbf{0}) = 0$.

$d := (d_1, \cdots, d_n)$. This column vector is a possible term structure, where $0 \leq d_j \leq 1$.[8] We let d^* be a term structure for given p, $b(\cdot)$, $T(\cdot)$, and A. It is any such d for which NA holds.

[7] See footnote 3.

[8] We assume that any investor has the option of paying any debt obligation in advance (without any reduction in the total amount of debt ultimately due). This completes the market in all but the last period and ensures that $d^* \leq 1$. Without this assumption, a much more complicated argument (appealing to the continuity of the model applied to the equivalent NA condition: $p - \beta \leq [A - \Omega]d \leq p + \beta$, where β is the n-vector of marginal transactions costs) is needed to show that the set of all no-arbitrage term structures is bounded.

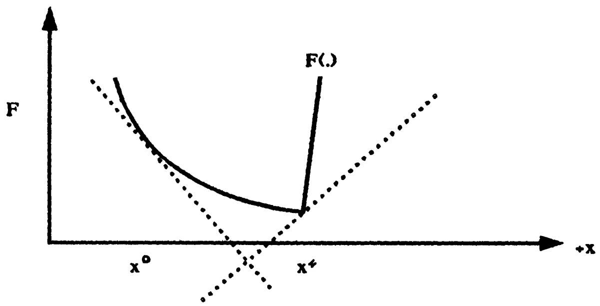

Figure 1. The slopes of the two dotted lines are two examples of subgradients of the function $F(\cdot)$ at the points where the lines touch $F(\cdot)$.

$p := (p_1, \cdots, p_m)$, which is the column vector of prices for the m riskless bonds traded in the market.

$T(\cdot)$ is the row n-vector function representing an investor's change in tax flow in future periods due to x. $T(\cdot) := [T_1(\cdot), \cdots, T_n(\cdot)]$, and $T(\mathbf{0}) = 0$. Each $T_j(\cdot)$ is a convex scalar function of x, as explained in Subsection IIB below.

B. Assumptions

(A1) (i) Total transactions costs, $b(x)$, is the sum of the transactions costs for each bond traded in x. $b(x)$ is paid currently.

(ii) For each bond, there is a particular set of flat or progressive piece-wise-linear transactions costs for long trades and another such set for short trades all with positive rates.[9]

(A2) For each type of income (subject to a distinct tax treatment), either there is a flat tax rate or there are progressive tax rates across income brackets.[10] Such rates are nonnegative and less than one hundred percent. There may be any finite number of types.

Assumption (A1ii) implies that the transactions-cost schedule for an investor's change of position, x_i, in any given bond i consists of a single rate or of progressive rates. This implies that the schedule for such change in any given bond is convex and piecewise linear in each x_i by definition of these properties. Then (A1i) implies that the schedule for such changes, $x = (x_1, \cdots, x_n)$, across all combinations of bonds is also both convex and piecewise linear in x since these properties are preserved under addition.

Assumption (A1ii) allows the transactions-cost rate for short sales to be zero for short sales up to the amount already held long. Note that the short-borrowing rate may be infinite for all shorting above some level; if this level is zero, shorting is prohibited. Since $b(\cdot)$ is convex and piecewise linear in x, our assumption that $b(\mathbf{0}) = 0$ ensures that $b(\cdot)$ has a kink (nondifferentiability) at the origin.

Assumption (A2) states that tax rates are progressive or flat, with or without exhaustible rebates for losses for each type of income. Hence, taxes are both

[9] See footnote 6.

[10] See footnote 4.

convex and piecewise linear in each type of income by the same reasoning used with $b(\cdot)$ immediately above. (A2) allows any number of types of income subject to distinct tax treatments (schedules), e.g., ordinary income, capital gain, non-taxable return of purchase price, and capital offset. Since convexity is preserved under addition and taxes are additive across tax treatments by definition of type of income, we see that taxes are convex in total pre-tax income, xA. This xA is obviously linear in x by construction. Since the composition of any convex function and any linear function is a convex function, we know that $T(\cdot)$ is convex in x.

Since $b(\cdot)$ and $T(\cdot)$ are convex and appear in the form $-b(\cdot)$ and $-T(\cdot)$ as the only nonlinear terms in $\Pi(\cdot)$ and $\Gamma(\cdot)$, respectively, they make $\Pi(\cdot)$ and $\Gamma(\cdot)$ concave in x. (See $b(\cdot)$ and $T(\cdot)$ in the objective, $\Pi(\cdot)$, and in the constraint, $\Gamma(\cdot)$, respectively, of (P).) Note that both $T(\cdot)$ and $b(\cdot)$ differ across investors and depend on their particular circumstances (e.g., previously acquired position).

III. Analysis

In Subsection IC, we define NA for this model in terms of the specific primal problem, (P), which is the maximization of riskless-arbitrage profit for the investor class facing the transactions cost schedule $b(\cdot)$ and the tax schedule $T(\cdot)$:

$$\max_x\{-xp - b(x)\} \quad \text{subject to } xA - T(x) \geq 0. \tag{P}$$

The objective function of (P) is the current cash proceeds of x net of transactions costs, i.e., $-xp - b(x)$. The constraint function is the future cash flow net of taxes.

Problem (D) below is dual to (P) above, as explained in Subsection IC, where A_j is the n-column-vector of payments from all bonds in period j:

$$\min_{d\geq 0}\left\{\max_x\left\{-xp - b(x) + \sum_{j=1}^{n} [xA_j - T_j(x)]\, d_j\right\}\right\}. \tag{D}$$

LEMMA 3.1: *WNA holds if and only if there is a term structure, $d^* \geq 0$, such that the optimal value of* (D) *vanishes. In other words, WNA holds if and only if there exists a $d^* \geq 0$ such that $(Ad^* - p)$ is a subgradient of the net-present-frictions function, $b(\cdot) + T(\cdot)d^*$, at $x = 0$, i.e., such that $x(Ad^* - p) \leq b(x) + T(x)d^*$ for all x.*

The proof is in the Appendix. Lemma 3.1 and the maximization of x in (D) imply that WNA is equivalent to $0 \geq -xp - b(x) + [xA - T(x)]d^*$ for all x and some $d^* \geq 0$. Hence, $x(Ad^* - p) \leq b(x) + T(x)d^*$ for all x, which means that the left-hand side of this "≤" never crosses above the right. Since the model assumes that $b(\mathbf{0}) = 0$ and $T(\mathbf{0}) = 0$, we see that the construction of (D) implies that both sides vanish, and hence touch, at $x = 0$. This touching without crossing defines the left-hand side as the graph of a supporting hyperplane of the right-hand side. It means that NA holds; i.e., there is no pre-friction profit, $x(Ad^* -$

p), in excess of the net present frictions, $b(x) + T(x)d^*$. This support is the relation between WNA and the exogenous circumstances, as per Figure 2.

Envision Schaefer's [19] model, SS, which has $b(\cdot) = 0$ and A as an after-tax payment matrix (so ignore $T(\cdot)$) instead of our before-tax matrix A. Then the net-present-frictions function in Figure 2 is just the horizontal axis. If $x(Ad - p)$ had any positive (negative) slope, then the arbitrage function, $x(Ad - p)$, would be positive for all positive (negative) x. Such slopes would provide infinite arbitrage. Hence, NA would imply that $Ad = p$. It is the nonzero transactions-cost function, $b(\cdot)$, that allows $Ad \neq p$ under NA and, thus (as we will show later), allows many term structures, d.

Assuming SNA instead of WNA in the hypothesis of Lemma 3.1 allows us to prove Lemma 3.1 for $d^* > 0$ instead of $d \geq 0$. This stronger hypothesis and conclusion is Lemma 3.2 in the Appendix, and it is used to prove the multiplicity result:

Result 3.1: Assume that CNA holds, or assume both that SNA holds and that the number of payment dates, n, is at least the number of bonds with linear independent marginal after-tax future cash streams. Then there is an infinite, convex, and closed set of no-arbitrage term structures, D, for each investor class.

Result 3.1 is proven in the Appendix. Here is an intuitive explanation of that proof. A linear function (the graph of which is a hyperplane), $x(Ad - p)$, supporting the net present frictions, $b(\cdot) + T(\cdot)d^*$, is a linear combination of constituent linear functions, $x\omega_0$ and $x\omega_j$, supporting $b(\cdot)$ and each $T_j(\cdot)$, respectively.[11] Figure 3 shows one such supporting hyperplane. This support is expressed mathematically as $x(Ad^* - p) \leq b(x) + T(x)d^*$ for all x. The weights of this linear combination are $(1, d_1, \cdots, d_n)$.

Since $b(\cdot)$ is kinked at the origin, it has an infinite number of supporting linear functions. The continuity in x of these functions supporting $b(\cdot)$ (without the $T(\cdot)d$) implies that it has a nonzero sweep angle through which it can rotate. That is, this (first) sweep angle can rotate clockwise and/or counterclockwise, sweeping out an infinite set of linear functions while continuing to support $b(\cdot)$ at the origin.[12]

Each of these linear functions supporting just $b(\cdot)$ at $x = 0$, in this first sweep, is a distinct constituent linear function, $x\omega_0$, and thus is a constituent of a distinct linear combination of linear functions supporting all of $b(\cdot) + T(\cdot)d$. Each such linear function supporting just $b(\cdot)$ at $x = 0$ provides a different combination. Hence, there is a supporting sweep angle for this linear combination, which is the same size as the first (larger if the tax rate changes at $x = 0$) but has a different position since it is rotated by $T(\cdot)d$. See Figure 3.

Since $b(\cdot)$, $T(\cdot)$, A, and p are exogenously fixed, each member of this infinite set of linear functions has a distinct combination, $(1, d_1, \cdots, d_n)$, and thus a distinct $d := (d_1, \cdots, d_n)$. Term structure is defined as any nonnegative $d =$

[11] See footnote 3.

[12] In the case of the hyperplane, $x(Ad - p)$, in Figure 2 having a segment coincident with a line segment of the frictions, the frictions will not immediately rotate faster than the hyperplane and thus cross below the hyperplane. This is because (A1) states that the rates of $T(\cdot)$ are always under one hundred percent.

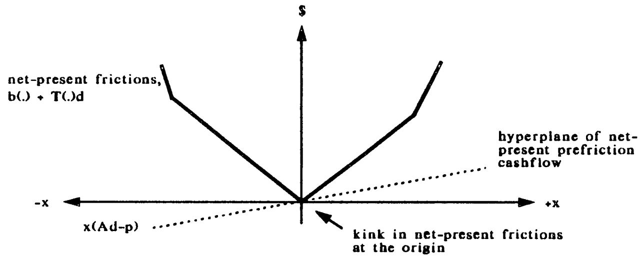

Figure 2. Assume that an investor has a given transactions-cost schedule, $b(\cdot)$, and tax schedule, $T(\cdot)$, as functions of his or her change of position, x. That a nonnegative n-vector d^* is a no-arbitrage term structure for this investor means that the present value of his or her (current and future) pre-friction cash flow for each x-value is not more than the frictions of that x-value, i.e., that $x(Ad^* - p) \leq b(x) + T(x)d^*$ for all x. This inequality is pictured as the dotted line being everywhere on or under the dark, solid, connected line segments representing the frictions function, $b(\cdot) + T(\cdot)d^*$.

$(d_1, \cdots, d_n)$ such that $x(Ad - p)$ supports $b(\cdot) + T(\cdot)d$ at $x = 0$. Lemma 3.2 in the Appendix shows that SNA is equivalent to the existence of such a d that is strictly positive. Hence, SNA is equivalent to the existence of at least one supporting linear function, $x(Ad^* - p)$, with $d^* > 0$, i.e., to at least one $d^* > 0$ being in the supporting sweep angle.

Let the angle swept out by these supporting functions as d varies through all strictly positive values be the positive-d sweep angle. Note that it contains $x(Ad^* - p)$, associated with the above $d^* > 0$ and drawn as a dotted line (as opposed to the dashed lines). Consider the shaded angle representing the intersection of the support sweep angle and this positive-d sweep angle. Since $x(Ad^* - p)$ is in this intersection, $x(Ad - p)$ is continuous in d, and $d^* > 0$, we see that this intersection is a nonzero angle and thus contains an infinite number of supporting linear functions, each with a distinct d^*.

IV. Schaefer's Example Revisited

A. Schaefer's [19] *Original Example, Which Has No Transactions Costs*

Two one-year bonds have \$100 face values and coupons of four and ten dollars, respectively. One class of investors is tax exempt and the other has a fifty percent ordinary rate and a twenty-five percent capital-gains rate. SS showed that WNA holds for the tax exempt if and only if $p_1/p_2 = 104d/110d = 104/110$, where d is the one-period discount factor. Similarly, WNA holds for the taxed class if and only if $p_1/p_2 = 77/80$. Since $77/80 \neq 104/110$, WNA cannot hold for both sets of investors and there is arbitrage if short sales are allowed. Obviously, the set of nonnegative d that will allow WNA is empty. Note that a capital-gains tax rate different from the ordinary-income tax rate is not needed for this paradox. A positive tax rate on income and no tax on the return of purchase price will suffice.

B. Schaefer's Example with Realistic Transactions Costs Added

Realistic transactions costs, $b(\cdot)$, are kinked (not differentiable) at $x = 0$; i.e., the rate does not approach zero as x approaches zero. Let ω_0 be a marginal

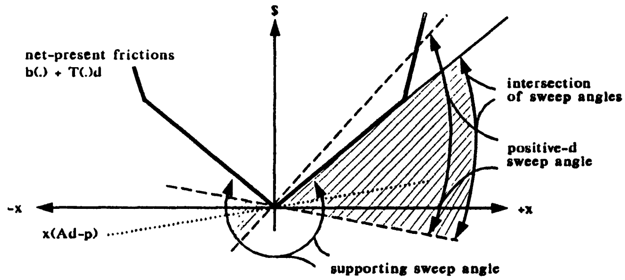

Figure 3. The set of no-arbitrage term structures is the infinite set of slopes in the intersection of the supporting sweep angle and the positive-d sweep angle. This intersection is represented by the hatched area.

current frictional cost, i.e., the slope of a linear function, $x\omega_0$, supporting $b(\cdot)$ at $x = 0$; thus, $x\omega_0 \leq b(x)$ for all x. This kink provides more degrees of freedom in the set of d^* that satisfies NA under these costs, i.e., satisfies $[A - \Omega]d = p + \omega_0$, where $[A - \Omega]$ is the after-tax version of A. Hence, there is a convex, closed, and uncountably infinite set of ω_0 that support the kink at the origin and thus satisfy NA, as explained in Section III.

This bounded "fat" set eliminates, within the extent of its magnitude, the arbitrage (per unit of x) that would otherwise stem from Ad differing from p. In doing so it provides an infinite set of d^* that satisfy NA. Thus, with sufficient "fatness" there is no need to prohibit short sales for NA to hold. Such a set of d^* is represented by the intersection of the sweep angles in Figure 3. In general, this set will differ across investor classes.

We introduce typical transactions costs, for a large investor with no previous position, into the original example in SS, where both bonds are held until they mature in one year. These costs were provided by major brokerage houses as proportions of face value: bid-ask spread of 1⁄32 of one percent (i.e., 0.0003125) and three percent annual borrowing cost for bonds sold short. Our investors do not lend bonds bought long in x to others for short borrowing because it is too difficult to short lend a bond for one year in real financial markets. Detailed descriptions of the transactions and typical rates involved in short borrowing a bond in a riskless buy-and-hold position are provided in Dermody and Rockafellar [3, 4].

With these costs, NA holds if and only if the inequality condition implying WNA in Lemma 3.1 holds. Simple algebra shows that, if $p_1 = \$98$ and $p_2 = \$102$, then this inequality condition, which is equivalent to WNA, holds for both sets of investors. The algebraic steps are in the Appendix to Section IV. The set of term structures for the tax exempt is $0.914035 \leq d \leq 0.927276$, a range of 132 basis points. For the taxed, it is $0.937817 \leq d \leq 0.965520$, a range of 277 basis points.

V. Minimum Transactions Costs

To facilitate our characterization of the minimum transactions costs necessary for NA, we will introduce some new notation. For each bond $i = 1, \cdots, m$ let β_i

be the marginal transactions cost of buying or selling associated with the bid-ask spread; let s_i be the short-borrowing cost. Thus, β and s are m-vectors that are specific to each investor class. The minimum current marginal transactions costs necessary to prevent WNA are the optimal solutions to the problem (PMIN):

$$\min_{\substack{\beta\geq 0\\ s\geq 0\\ d\geq 0}} \sum_{i=1}^{m} (\beta_i + s_i) \quad \text{subject to} \quad p - \beta - s \leq [A - \Omega]d \leq p + \beta, \qquad \text{(PMIN)}$$

where $[A - \Omega]$ is the marginal after-tax payment matrix for the investor class in question. The dual of this problem, (DMIN) below, is the maximization of arbitrage in the absence of transactions costs, where the investor's change of position, x, is restricted to x such that $|x_i| \leq 1$ for all bonds i:

$$\max_{x} - xp \quad \text{subject to} \quad x[(A - \Omega)] \geq \mathbf{0},$$

$$\text{and} \quad |x_i| \leq 1 \quad \text{for} \quad i = 1, \cdots, m. \qquad \text{(DMIN)}$$

These primal and dual problems are linear programs. Since they are feasible ($x = 0$ solves the dual) and (PMIN) is bounded by its nonnegativity constraint, the two problems must have common finite optimal values. They also must have finite optimal solutions, β^* and s^* for (DMIN) and x^* for (PMIN). These β^* and s^* are the minimal transactions costs we seek, and they are expressed in (PMIN) in terms of the after-tax payment matrix, $[A - \Omega]$, and p. The optimal value of (DMIN), say $(DMIN)^*$, is the maximum arbitrage for a type of normalized portfolio in a market without transactions costs, which is the minimum total transactions cost for short sales, $\beta + s$, necessary for NA. Our normalized portfolio is one for which the magnitude, $|x_i|$, of the change in position, x_i, for each bond i is not more than one unit of that bond. This is a very intuitive result that we will use in the next section:

Result 5.1: Consider an investor in a market without transactions costs. The sum, $\sum(\beta_i + s_i)$, of the minimum transactions costs that would have to be imposed in order to avoid arbitrage is the maximum current arbitrage available when the investor is restricted to portfolios that have a magnitude of at most one unit in each bond in the market.

VI. Clientele Effects

Starting with the notation of Section V, we turn to the clientele effects in a market with the realistic form of transactions costs we are treating in this paper. Assume β and s sufficient to ensure WNA. We will also use some notation, results, and the approach of Dermody and Rockafellar [4, 5]. Let X_i be the change of position (of the investor in question) involving an increase in his or her long position in bond i, and let x_i be that involving an increase in his or her short position in bond i. Thus, X_i and x_i are non-negative.

To the investor class in question, the price of each bond i is bounded by the upper and lower value described here because WNA holds. The upper value is the minimum current cash that the investor must spend to replicate future cash flow at least equal to that of bond i, which is the n-vector $A_i - T(e_i)$. It is the optimal value of

$$\min_{\substack{X\geq 0\\ X\geq 0}}[(X(p+\beta) - x(p-\beta-s)] \quad \text{subject to}$$

$$(X-x)A - T(X-x) \geq A_i - T(e_i), \qquad (\mathrm{U_i})$$

where $e_i := (0, \cdots, 0, 1, 0, \cdots, 0)$ has a one in the ith position. Thus, $T(e_i)$ is the change in tax flow stemming from longing one unit of bond i for the investor in question. We call the optimal value of this problem $(U_i)^*$.

The lower value is the maximum current cash that the investor can extract from the market in exchange for obligating the future after-tax cash flow of bond i. It is the optimal value of

$$\max_{\substack{X\geq 0\\ X\geq 0}}\, [x(p-\beta-s) - X(p+\beta)] \quad \text{subject to}$$

$$(X-x)A - T(X-x) \geq -[A_i - T(e_i)]. \qquad (\mathrm{L_i})$$

We call the optimal value of this problem $(L_i)^*$. It is the maximum current cash the investor can obtain given an after-tax subsidy to future cash flow of $A_i - T(e_i)$. This subsidy allows him or her to make a trade (X, x) that will produce at least the negative of the after-tax cash flow of bond i in each future period j instead of producing at least 0. See Dermody and Rockafellar [4, 5] for a more complete analysis.

Suppose the investor could long bond i for less than $(L_i)^*$. Then, the investor would use the future cash flow of bond i as a subsidy to his or her future cash flow in order to obtain current after-tax cash equal to $(L_i)^*$, which is more than he or she pays for bond i, thus violating WNA. Thus, WNA implies that $p_i + \beta_i \geq (L_i)^*$. Symmetrically, WNA implies that $(U_i)^* \geq p_i - \beta_i - s_i$.

The clientele effects in our model are different from clientele effects in the usual sense because an investor may be unwilling to buy a bond and at the same time be unwilling to short the same bond, even if he or she has previously acquired it (and thus shorting is just selling). Furthermore, an investor neither long nor short for a given bond i is the case, in general, under CNA. We refer to the investor not buying bond i with no condition on his or her shorting it as a weak long clientele effect for bond i. We refer to the symmetric condition of an investor not shorting bond i with no condition on his or her longing it as a weak short clientele effect for bond i. In contrast, the usual clientele effect is an investor's not buying a bond and shorting that bond if not prohibited, and in particular selling any he or she already holds. We will call this strong long clientele. We symmetrically define strong short clientele.

Since an investor could get the amount of current cash equal to $(L_i)^*$ for the cash flow of bond i, he or she would not short it for less. (I.e., if $p_i - \beta_i - s_i < (L_i)^*$, he or she would not short it.) Said another way, since $(L_i)^*$ is the maximum cash from obligating A_i, and since shorting bond i itself is one way to obligate A_i, the net short price, $p_i - \beta_i - s_i$, must not be above $(L_i)^*$ by definition of $(L_i)^*$.

Symmetrically, the net long price, $p_i + \beta_i$, is not below $(U_i)^*$ by definition.[13] Some investor must be buying each bond bought and someone must be selling. Thus, there is some investor with a d^* such that $[A_i - T(e_i)]d^* \geq p_i + \beta_i$, and another with a d^* such that $[A_i - T(e_i)]d^* \leq p_i - \beta_i - s_i$ for each bond traded in the market. WNA ensures that these last two weak inequalities hold as equalities.

Linear programming tells us that $(U_i)^* = \max_{d \in D} dw$, and $(L_i)^* = \min_{d \in D} dw$, where D is the set of all no-arbitrage term structures for the investor class in question, i.e., for the frictions facing that class. Dermody and Rockafellar [4, Theorem 5.3] show that D has a nonempty interior under CNA. Hence, CNA implies that $(L_i)^* < (U_i)^*$ by the standard results of linear programming. This provides our clientele story, which is illustrated in Figure 4 below and stated formally in Result 6.1.

Result 6.1: Complete no-arbitrage, CNA, implies that, for each bond $i = 1, \cdots, m$,

$$p_i - \beta_i - s_i \leq (L_i)^* < (U_i)^* \leq p_i + \beta_i.$$

Here, $(p_i - \beta_i - s_i)$ and $(p_i + \beta_i)$ are the net short and long prices, respectively, and $(L_i)^*$ and $(U_i)^*$ are the minimum and maximum values, respectively, of bond i to the investor class in question.

For each bond i, each investor has a weak long clientele effect or values bond i equal to $(L_i)^* = p_i - \beta_i - s_i$, but not both, and each investor has a weak long clientele effect or values bond i equal to $(U_i)^* = p_i + \beta_i$, but not both. Hence, for each bond i, each investor has a weak short and/or a weak long clientele effect since he or she cannot value it equal to both $p_i - \beta_i - s_i$ and $p_i + \beta_i$.

COROLLARY 6.1: *Complete no-arbitrage, CNA, implies that, for any given traded bond i, there is at least one investor (in some class) with a strong long clientele effect for it, and another investor with a strong short clientele effect for it.*

Proof: At least one investor must be longing bond i, which implies under WNA that there is a $d^* > 0$ such that he or she values bond i as $[A_i - T(e_i)]d^* = p_i + \beta_i$, which is the net long price. However, the net short price, $p_i - \beta_i - s_i$, is less than $p_i + \beta_i$, which implies that this investor values bond i above the net short price. Hence, he or she has a strong short clientele effect for bond i. The symmetric argument shows that another investor must have a strong long clientele effect for bond i. Q.E.D.

For each investor class, the set of no-arbitrage term structures, D, is a distinct n-dimensional polyhedron with a nonempty (i.e., fat in all dimensions) interior in the n-space of discount factors or term structures, $d = (d_1, \cdots, d_n)$. (See the example of the very large D for realistic transactions costs in Subsection IVB.) This D is defined by, at most, m pairs of parallel faces, where each face is an $(n-1)$-dimensional hyperplane in the n-space of term structures. For each bond i, one face of the pair consists of the nonnegative d-vectors that would value the future cash flow of bond i (as $A_i d$) equal to the net short price, $p_i - \beta_i - s_i$, of

[13] See footnote 6.

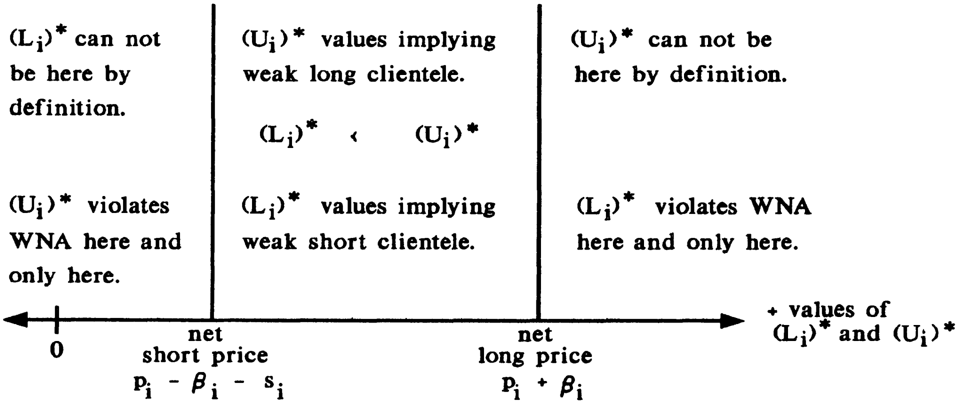

Figure 4. $(U_i)^*$ being more than the net long price of bond i violates the definition of $(U_i)^*$. If $(U_i)^*$ is strictly between (and not equal to either of) the net short and net long prices, then all investors in the class in question would have a weak long clientele effect for bond i. $(U_i)^*$ being less than the net short price violates weak no-arbitrage. Complete no-arbitrage, CNA, implies that $(U_i)^*$ is more than the net short price. The symmetric statements hold for $(L_i)^*$ with the words "long" and "short" reversed.

If $(L_i)^*$ equals the net short price, then there may be some investors in the class who exhibit the usual (strong long) clientele effect for bond i. Such an investor would short bond i (sell it if he already owned it) and would not long it. His or her personal d^* would value bond i equal to $(L_i)^*$. No other investor would exhibit the usual clientele effect. If $(U_i)^*$ equals the net long price, then there may be some investors in the class who have a strong short clientele effect for bond i.

bond i. The other face consists of such d for which $A_i d$ equals the net long price, $p_i - \beta_i$, of bond i. This set D is a conveyer of information about the investor class. (See Dermody and Rockafellar [4] for the formal development of this theory.)

In particular, D tells us about weak clientele. The face associated with being long in bond i does not touch D if and only if every investor in the investor class has a weak long clientele effect for bond i. Similarly, the face associated with shorting bond i does not touch D if and only if every investor in the class has a weak short clientele effect for bond i.

Since we will consider a bond i only if it is traded, there is at least one investor class whose D touches the long hyperplane associated with bond i. At the point where they touch, $(U_i)^* = [A_i - T(e_i)]d^* = p_i + \beta_i$, which is the net long price as shown in Figure 4. Similarly, every traded bond must be sold (shorted if $s_i > 0$ for the class selling), which implies that there is at least one investor class whose D touches its short hyperplane associated with bond i. At that point, $(L_i)^* = [A_i - T(e_i)]d^* = p_i - \beta_i - s_i$. The geometry of D for each class implies that strong long clientele is equivalent to the hyperplane associated with being long in bond i, being redundant in defining D. The geometry is symmetrical for weak short clientele. For a fixed arbitrary investor among many, we expect in general to have $(L_i)^* < [A_i - T(e_i)]d^* < (U_i)^*$, which implies both weak long and weak short clientele for each bond i.

We have shown that there is a large set of no-arbitrage term structures for each investor class. A natural question arises as to how this analysis relates to the traditional views. These traditional views are that an individual investor class always has a unique no-arbitrage term structure and that there is a unique pricing operator for all future cash flows. This would be a reasonable question for the case of an individual investor (in a given class) whose intertemporal-utility function is differentiable and known to us. His or her unique personal term structure could value the future cash flow of bond i equal to one of, but obviously not both of, the vertical lines in Figure 4. Hence, he or she would exhibit a weak long and/or a weak short clientele effect for each bond i, as is the case in Result 6.1.

In general, such a unique term structure, d^*, would value bond i between the two lines in Figure 4, and, thus, the investor would have both long and short weak clientele effects. In the absence of such a known utility function, the entire "fat" set D is the pricing operator, and, for each future cash flow, there is a distinct corner of D, d^*, that is associated with $(U_i)^*$ and another associated with $(L_i)^*$, as explained in Dermody and Rockafellar [3, 4].

Appendix to Section III

Let $\Omega = (\omega_1, \cdots, \omega_n)$ be an $m \times n$ matrix of subgradients of $T(\cdot)$ at $x = \mathbf{0}$. That is, ω_i and $\mathbf{0}$ are m-dimensional vectors and ω_i is in the set, $\partial T_j(\mathbf{0})$, of subgradients of T_j at $x = 0$, for $j = 1, \cdots, n$. Let ∂B be the set of all (m-vector) subgradients of the current transactions-cost function, $b(\cdot)$, at $x = \mathbf{0}$; let ω_0 be an m-dimensional vector in ∂B. Define $\partial_p B$ as ∂B translated by p, i.e., as the m-dimensional set $p + \partial B$. We restate Lemma 3.1 mathematically.

LEMMA 3.1: *WNA holds if and only if there exists a term structure, $d^* \geq 0$, such that $(Ad^* - p)$ is a subgradient of the frictions function, $b(\cdot) + T(\cdot)d^*$, at $x = 0$, i.e., such that $[A - \Omega]d^* = p + \omega_0$.*

Proof: WNA is defined as $(P)^* = 0$. The convexity of the objective and constraints in x implies, via the duality theory of convex programming, that $(P)^* = 0$ if and only if the optimal value of (D), $(D)^*$, equals 0 and both have finite optimal solutions. Since (D) is a min over d and a max over x, we have $(D)^* = 0$ if and only if (1) holds for some $d^* \geq 0$. Hence, WNA holds if and only if (1) holds for some $d^* \geq 0$, and, by simple algebra, (1) holds if and only if (2) holds for the same d^*:

$$-xp - b(x) + \sum_{j=1}^{n} [xA_j + T_j(x)\ d_j]^* \leq 0 \quad \text{for all } x, \tag{1}$$

$$x[Ad - p] \leq b(x) + \sum_{j=1}^{n} d_j T_j^*(x) \quad \text{for all } x. \tag{2}$$

Both sides of (2) vanish at $x^* = 0$. This and (2) holding for all x is by definition the statement that there is a $d^* \geq 0$ such that $(Ad^* - p)$ is a subgradient of $b(\cdot) + T(\cdot)d^*$ at $x = 0$; i.e., $(Ad^* - p)$ is in $\partial(b + Td^*)(0)$ for some $d^* \geq 0$, where $\partial(b + Td^*)(0)$ is the set of all subgradients of the net-present-frictions function at the origin. Hence, WNA is equivalent to a $d^* \geq 0$ such that $[A - \Omega]d^* = p + \omega_0$. Q.E.D.

Let $I(\Omega)$ be the image of all $d > \mathbf{0}$ under the linear transformation $[A - \Omega]$ for any fixed arbitrary Ω in $\partial T(\mathbf{0})$. That is, $I(\Omega) = \{z \text{ in } R^m \mid \exists d > \mathbf{0} \text{ such that } [A - \Omega]d = z\}$. This leads directly to Lemma 4.2, which is slightly stronger than Lemma 4.1.

LEMMA 3.2: *SNA holds if and only if Lemma* 3.1 *holds for* $d^* > 0$, *i.e., if and only if* $I(\Omega) \cap \partial_p B$ *is nonempty.*

The proof is in Prisman [12, Subsection IV*C*].

Proof of Result 3.1: Lemma 3.2 and our hypothesis of CNA imply that there exist $d^* > \mathbf{0}$ and a Ω^* in $\partial T(\mathbf{0})$ such that $z^* = [A - \Omega]d^*$ and z^* is in $\partial_p B$. The two cases extant here depend on the number of payment dates, n, being as large or not as large as the number of bonds with linearly independent after-tax cash flows, i.e., on row rank of $[A - \Omega]$.

Case i: $[A - \Omega^*]$ *has full row rank of m.* Only in this case must we assume CNA. CNA implies, via Theorem 27.1(d) in Rockafellar [14], that $[A - \Omega]$ has full dimension m. By construction, $I(\Omega)$ is open. Hence, there exists $\delta > 0$ such that the m-dimensional neighborhood $N_\delta(z^*) := \{z \,\|\, z - z^* \| < \delta\}$ is contained in $I(\Omega^*)$. Since $I(\Omega^*)$ is open and both $I(\Omega^*)$ and $N_\delta(z^*)$ have full dimension, m, we see that $\Psi(\delta) := [N_\delta(z^*) \cap \partial_p B]$ is not empty and contains uncountably infinite points. By definition, z is in $\Psi(\delta)$ if and only if z is in $I(\Omega^*)$ and $\exists\ \omega_0 \in \partial_p B$ such that $z = \omega_0 + p$.

Then $\Psi^{-1}(\delta) := \{d \mid [A - \Omega^*]d = z,\ z \in \Psi(\delta)\}$ contains uncountably infinite points, each of which satisfies the condition in Lemma 3.2, i.e., satisfies that $[I(\Omega^*) \cap \partial_p B]$ is nonempty. Thus, $\Psi^{-1}(\delta)$ is in the set of no-arbitrage term structures, D, by definition. $\Psi(\delta)$ is convex because both $\partial_p B$ and $N_\delta(z^*)$ are. Hence, $\Psi^{-1}(\delta)$ is convex. The "=" in the construction of $I(\Omega)$, which itself appears in the construction of $\Psi(\delta)$, and the continuity of our model imply that $\Psi(\delta)$ is closed and, thus, that $\Psi^{-1}(\delta)$ is closed. Case i is done.

Case ii: $[A - \Omega^*]$ *does not have full row rank of m.* This case implies that the nullity of $[A - \Omega]$ is positive and, thus, that there is another d, say d^{**}, such that $[A - \Omega]d^{**} = p - \omega_0$. By simple algebra, this holds for all convex combinations of d^* and d^{**}, say d^c. By definition, such d^c are in D. Since this equality holds for all such d^c, the set of all such d^c is convex. Since $d^* > 0$, there is an uncountably infinite and convex set of such $d^c > 0$. The linearity and continuity of the last equality imply that this set is closed. Q.E.D.

Appendix to Section IV

Lemma 4.1 implies that $x(Ad - p) \leq b(x) + T(x)d^*$ for all x. For tax-exempt investors in our example, this means that $(x_1, x_2)[(104d - p_1), (110d - p_2)] \leq b(x)$ for all $x \in R^2$, where $b(x) = 0.0003125(|x_1| + |x_2|) + 0.03[-\min\{0, x_1\}p_1 - \min\{0, x_2\}p_2]$. For taxed investors, this means that $(x_1, x_2)[\{4(1 - 0.5) + (100 - p_1)(1 - 0.25) + p_1\}d - p_1, \{10(1 - 0.5) + (100 - p_2)(1 - 0.25) + p_2\}d - p_2] \leq b(x)$ for all $x \in R^2$ and the same $b(x)$. Both sides of these "$\leq$" are positively homogenous of degree one. Hence, each side holds for any given d and all x if and only if it holds for that d and the orthonormal long and short x-values: (1, 0), (−1, 0), (0, 1), and (0, −1).

For the untaxed investor, the above implies that (i) $p_1 + 0.0003125 \geq 104d$ and $p_2 + 0.0003125 \geq 110d$ and (ii) $p_1 - 0.03p_1 - 0.0003125 \leq 104d$ and $p_2 - 0.03p_2 - 0.0003125 \leq 110d$. For taxed investors, the above implies that (i) $p_1 + 0.0003125 \geq [4(1 - 0.5) + (100 - p_1)(1 - 0.25) + p_1]d$ and $p_2 + 0.0003125 \geq [10(1 - 0.5) + (100 - p_2)(1 - 0.25) + p_2]d$ and (ii) $p_1 - 0.03p_1 - 0.0003125 \leq [4(1 - 0.5) + (100 - p_1)(1 - 0.25) + p_1]d$ and $p_2 - 0.03p_2 - 0.0003125 \leq [10(1 - 0.5) + (100 - p_2)(1 - 0.25) + p_2]d$. Conditions (i) and (ii) prevent arbitrage via longing and shorting, respectively, i.e., via a bond violating one of the "≥" and "≤" conditions, respectively.

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